

# VALVE CLOSURE IN GRAPH-THEORETICAL MODELS FOR SLOW TRANSIENT NETWORK ANALYSIS

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**ABSTRACT:** A valve closure algorithm is presented for inclusion within a slow transient (rigid water column) pipe network model. The algorithm is specifically formulated for (but not limited to) node-based, graph-theoretical models and is both direct and accurate. It is distinguished from conventional approaches by its direct assignment of the final discharge at closure to zero and subsequent use of the head loss at the closing valve to compute the flow rate in pipes incident to the closing valve. As a result, redistribution of residual flow prior to closure to these incident pipes is not required, leading to a less computationally demanding and more robust algorithm than previously published valve closure procedures. Application of the valve closure algorithm is illustrated in a pipe network, which also nicely demonstrates again the relation between rigid water column and water hammer models.

## INTRODUCTION

Unsteady flows in pipe networks can occur for many reasons, including accumulation at reservoirs, adjustments in demand, or change in the operating condition of pumps. One important event leading to this unsteadiness in a pipe network is the opening or closing of a valve. Altering the rate of flow through a pipe often causes an immediate change of pressure. When these changes in flow take place relatively slowly, the compressibility or elasticity of the fluid can be neglected (Pickford 1969). It is under these conditions that a lumped or rigid water column network approach finds utility, since a much larger time step can be used than in a water hammer analysis (Abreu et al. 1991). Unlike time-stepping steady state approaches (such as EPANET), rigid water column models account for the inertia of the water column.

When a valve closes from an almost closed position to a fully closed position, the flow  $Q$  through the valve is reduced from some finite value to zero, a reduction that is fundamentally incompatible with a rigid water column approach. Shimada (1989) and others noted that this final closure often causes considerable numerical difficulty in a rigid water column model. Specifically, a discontinuity occurs at nodes upstream and downstream of the valve. Since it is unlikely that the explicit finite difference prediction  $\Delta t \cdot dQ/dt$  will be equal to the actual  $\Delta Q$  associated with the change in position from almost closed to fully closed, the flow in the numerical solution is often arbitrarily reduced from some finite value to zero. However, this flow is reduced in isolation from other pipes in the network, so the correct flow rate through the valve will not be reflected in pipes incident to the valve's upstream and downstream nodes. Thus, mass is not conserved at these nodes.

To avoid this problem, but to salvage the practical utility of the larger time step associated with the rigid water column approach, Shimada (1989) proposed a correction technique in which the water mass is distributed to pipes incident to the end nodes of the valve. The discharge is reduced to zero when the flow through the closing valve is less than or equal to  $0.01 \text{ m}^3/\text{s}$ . A correction of  $0.01 \text{ m}^3/\text{s}$  is carried throughout the net-

work by following pipe paths incident to these nodes. However, the method is arbitrary, iterative, and computationally expensive.

The valve closure approach presented in this paper preserves an accurate mass balance at nodes incident with the valve because the flow is computed directly using an appropriate head loss at the closing valve. The head loss is calculated using the effective deceleration rate  $\Delta Q/\Delta t$  over the time step preceding closure. Flow through the closing valve is no longer arbitrarily reduced to zero, making the method less sensitive to both the magnitude of flow at the almost closed position and the time step. The algorithm presented herein is first-order implicit and numerically stable, even for discharges less than  $0.01 \text{ m}^3/\text{s}$ . In addition, it is not constrained by the size of the time step and does not require iteration unless two or more valves in the network close during the same time step.

## VALVE CLOSURE ALGORITHM

To demonstrate the valve closure algorithm, consider the water distribution network shown in Fig. 1. Roman numerals indicate pipe or edge indices while Arabic numerals indicate pipe junctions or nodes. A supply reservoir delivers water to a terminal reservoir by gravity flow. Two storage tanks connect to the network at 3.5 and 7.5 km downstream from the supply reservoir. A circular gate valve (discharge coefficient  $E_v = Q_v/\sqrt{H_v} = 1.0 \text{ m}^{5/2}/\text{s}$ , where  $Q_v$  = discharge through the valve and  $H_v$  = valve head loss) regulates flow into the terminal reservoir and is opened or closed according to a parabolic  $\tau$ -curve described in symbols as follows (Karney and Ruus 1985):  $\tau = [(2s - s^2)/0.6377][1 - 0.3623(2s - s^2)]$ , where  $\tau$  = relative effective area of the valve opening;  $s = 1 - t/T_c$  = relative residual valve travel;  $t$  = time since beginning of closure; and  $T_c$  = time of closure. Nodes 3 and 6 represent the supply and terminal reservoirs, respectively, and nodes 4 and 5 are storage tanks. This water distribution system intentionally resembles the example in Shimada (1989) so as to conveniently compare the proposed valve closure algorithm to Shimada's approach.

Assume the valve upstream of the terminal reservoir is fully closed at time  $t$ . If the flow through the valve is  $Q_v(t - \Delta t)$

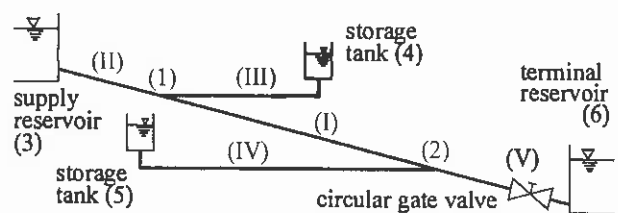


FIG. 1. Network with Storage Tanks

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when the valve is almost closed, a first-order approximation of the rate of change of the discharge over this time interval (i.e.,  $\Delta t$ ) is

$$\frac{\Delta Q_v}{\Delta t} = \frac{0 - Q_v(t - \Delta t)}{\Delta t} = -\frac{Q_v(t - \Delta t)}{\Delta t} \quad (1)$$

This paper simply approximates  $dQ_v/dt$  at time  $t$  (i.e., when the valve is fully closed) by  $\Delta Q_v/\Delta t$ , also a common assumption when modeling dynamic check valve behavior (Thorley 1989). Having assigned  $dQ_v/dt$ , the head difference in edge V associated with the valve's closure can be obtained. Although this approach can be applied to any network solver, it is illustrated here using Shimada's approach.

As derived in detail by Shimada (1989), the node-based, graph-theoretical system equation for lumped analysis of a pipe network is rewritten here as

$$\frac{dQ}{dt} = W \cdot M - Y \quad (2)$$

where  $dQ/dt$  = rate of change of flow in the various pipes;  $W$  = known coefficient matrix in part representing pipe inertia;  $M$  = head difference in each pipe; and  $Y$  = known rate of change of outflow from various nodes. To be complete, a few additional details are required to illustrate the new approach. Specifically,

$$M = G_1^T \cdot H_1 + G_3^T \cdot H_3 - F(Q) - V(Q) \quad (3)$$

$$Y = L^{-1} \cdot G_2^T \cdot R^{-1} \cdot \frac{dq_2(t)}{dt} \quad (4)$$

where  $G_1^T$  = transpose of incidence matrix rows corresponding to nodes having both an unknown head and unknown outflow (e.g., storage tanks, surge tanks, air chambers, and air-vacuum valves) with  $H_1$  being the corresponding nodal head vector;  $G_3^T$  = transpose of incidence matrix rows corresponding to nodes having a known head and an unknown outflow (e.g., constant-head reservoirs) with  $H_3$  being the corresponding nodal head vector;  $F(Q)$  = pipe wall friction loss vector;  $V(Q)$  = valve loss vector;  $G_2^T$  = transpose of incidence matrix rows corresponding to nodes having an unknown head and a known outflow (e.g., a multipipe junction) with  $q_2$  = corresponding nodal outflow vector; and  $R, L$  = known coefficient matrices representing pipe inertia in its various forms [see Shimada (1989) for details].

The vector  $M$  represents the head difference in each edge. For the pipeline system in Fig. 1

$$\begin{pmatrix} Q'_i \\ Q''_i \\ Q'''_i \\ Q'_{iv} \\ Q''_{iv} \end{pmatrix} = \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \\ \text{V} \end{matrix} \begin{pmatrix} W_{i,I} & W_{i,II} & W_{i,III} & W_{i,IV} & W_{i,V} \\ W_{ii,I} & W_{ii,II} & W_{ii,III} & W_{ii,IV} & W_{ii,V} \\ W_{iii,I} & W_{iii,II} & W_{iii,III} & W_{iii,IV} & W_{iii,V} \\ W_{iv,I} & W_{iv,II} & W_{iv,III} & W_{iv,IV} & W_{iv,V} \\ W_{v,I} & W_{v,II} & W_{v,III} & W_{v,IV} & W_{v,V} \end{pmatrix} \begin{pmatrix} M_i \\ M_{ii} \\ M_{iii} \\ M_{iv} \\ M_v \end{pmatrix} - \begin{pmatrix} Y_i \\ Y_{ii} \\ Y_{iii} \\ Y_{iv} \\ Y_v \end{pmatrix} \quad (5)$$

where  $Q'_i = dQ_i/dt$ ;  $W$  = individual element inertia terms;  $M$  = individual pipe head difference terms; and  $Y$  = individual demand acceleration terms. The four unknowns in (5) are  $Q'_i, Q''_i, Q'''_i, Q'_{iv}$ , and  $M_v$ . However,  $M_v$  is obtained independent of and prior to the solution of  $Q'_i$  through  $Q'_{iv}$  if the matrix operations associated with the fifth row in (5) are performed. To demonstrate more clearly the process of matrix contraction, the fifth row of (5) is written separately as

$$-\frac{Q_v(t - \Delta t)}{\Delta t} = W_{v,I} \cdot M_i + W_{v,II} \cdot M_{ii} + W_{v,III} \cdot M_{iii} + W_{v,IV} \cdot M_{iv} + W_{v,V} \cdot M_v - Y_v \quad (6)$$

where the value of  $Q'_v$  from (1) is used. Isolating  $M_v$  in (6) gives the head difference upon valve closure in edge V as

$$M_v = \frac{-\frac{Q_v(t - \Delta t)}{\Delta t} + Y_v - W_{v,I} \cdot M_i - W_{v,II} \cdot M_{ii} - W_{v,III} \cdot M_{iii} - W_{v,IV} \cdot M_{iv}}{W_{v,V}} \quad (7)$$

Substituting the right-hand side of (7) for  $M_v$  in (5) and solving gives the actual value of  $Q'_i$  through  $Q'_{iv}$  when the valve in edge V fully closes. Each of the first-order differential equations in (5) is integrated to give accurate values of  $Q_i$  and  $Q_{iv}$  at time  $t$ . Of course,  $Q_v = 0$  at time  $t$ .

## SIMULTANEOUS VALVE CLOSURE ALGORITHM

As stated earlier, this algorithm requires iteration if two or more valves close simultaneously. By "simultaneously," it is meant that each valve attains a fully closed position during the same time step. (Iteration is not required if two or more valves are simply closing, but do not attain a fully closed position simultaneously.) To demonstrate the iterative procedure, consider once again the pipe network in Fig. 1, but assume a second valve has been installed in edge III and is closed in exactly the same manner as the valve in edge V. Thus, (1) still holds with an exactly analogous expression for edge III. In this case, both the head difference in edge III and the head difference in edge V are unknown. However,  $M_{iii}$  and  $M_v$  can be obtained independent of and prior to the solution of  $Q'_i, Q''_i, Q'''_i, Q'_{iv}$ . From (7),  $M_{iii}$  and  $M_v$  are defined as follows:

$$M_{iii} = \frac{-\frac{Q_{iii}(t - \Delta t)}{\Delta t} + Y_{iii} - \sum_{i=1}^n W_{iii,i} \cdot M_i - \sum_{j=IV}^v W_{iii,j} \cdot M_j}{W_{iii,iii}} \quad (8)$$

$$M_v = \frac{-\frac{Q_v(t - \Delta t)}{\Delta t} + Y_v - \sum_{i=1}^{iv} W_{v,i} \cdot M_i}{W_{v,v}} \quad (9)$$

Eqs. (8) and (9) represent a system of two equations and two unknowns that can be solved by simple substitution of (9) into (8), by iteration, or by using matrix methods. The iteration method is surprisingly efficient and effective, particularly for larger systems.

Initial estimates of  $M_{iii}$  and  $M_v$  can be obtained from the conditions at time  $t - \Delta t$ . These initial estimates are often good; therefore, a simple head difference correction technique that has been shown to rapidly converge to the exact solution is proposed in place of the more conventional Newton-Raphson approach to solving a system of equations. Using  $M_v(t - \Delta t)$  and  $M_{iii}(t - \Delta t)$ , (8) is solved for  $M_{iii}(t)$  and (9) is solved for  $M_v(t)$ . If either  $M_{iii}(t)$  or  $M_v(t)$  differ from their initial estimates, (8) and (9) are re-solved using  $M_{iii}(t)$  and  $M_v(t)$  as initial estimates of  $M_{iii}$  and  $M_v$ . This procedure is repeated until the initial estimates differ from the corresponding solutions of (8) and (9) by less than a specified tolerance  $\epsilon_v$ . A limit of  $10^{-7}$  is generally suitable when working in SI units. For systems with two or three valves closing simultaneously, only two iterations are often required. Details of this valve closure procedure are summarized in pseudocode in Fig. 2. The numerical performance of this algorithm is illustrated next.

## NUMERICAL EXAMPLE

A full valve closure in the water distribution network shown in Fig. 1 is presented. This numerical example is based on an

1. Calculate  $R$ ,  $R^{-1}$ , and  $W$ .
2. For  $i = 1, \dots, V$ 
  - If  $\tau_i(t) = 0$  and  $\tau_i(t - \Delta t) \neq 0$ ,
    - ◊ Set  $M'_i = M_i(t - \Delta t)$ .
  - else
    - ◊ Set  $M'_i = M_i(t)$ .
3. Set  $k = 0$ .
4. For  $i = 1, \dots, V$ 
  - If  $\tau_i(t) = 0$  and  $\tau_i(t - \Delta t) \neq 0$ ,
    - ◊ Calculate
$$M_i(t) = -\frac{Q_i(t - \Delta t)}{\Delta t} + Y_i - \sum_{j=1}^{i-1} W_{i,j} \cdot M'_j - \sum_{j=i+1}^V W_{i,j} \cdot M'_j$$
    - If  $|M_i(t) - M'_i| > \epsilon_{v,i}$ ,
      - ◊ Set  $M'_i = M_i(t)$ .
      - ◊ Set  $k = k + 1$ .
5. If  $k > 0$ , goto 2.
6. Calculate  $\frac{dQ_I}{dt}$ ,  $\frac{dQ_{II}}{dt}$ ,  $\frac{dQ_{IV}}{dt}$ ,  $Q_I$ ,  $Q_{II}$ , and  $Q_{IV}$ .

FIG. 2. Pseudocode for Simultaneous Valve Closure in Pipe Network

TABLE 1. Network Dimensions, Characteristics, and Steady State Edge Flows

Edge (1)	Length (m) (2)	Diameter (mm) (3)	Friction factor $f_p$ (4)	Wavespeed (m/s) (5)	Flow (m <sup>3</sup> /s) (6)
I	3,500	1,200	0.012	1,010	1.97
II	4,000	1,500	0.010	1,026	1.97
III	1,500	1,000	0.010	1,154	0.00
IV	1,300	800	0.012	1,000	0.00
V	2,200	1,000	0.010	1,015	1.97

earlier experiment by Axworthy and Karney (1997). Table 1 shows the dimensions and properties of each pipe in Fig. 1. The steady state nodal pressure heads are as follows: node 1, 28.3 m; node 2, 22.9 m; node 3, 30.0 m; node 4, 28.3 m; node 5, 22.9 m; and node 6, 12.0 m. The valve is closed according to the parabolic relation introduced earlier. The results of a lumped analysis, using the valve closure algorithm, are compared to the results of a water hammer analysis for the purpose of demonstrating model consistency. The lumped analysis was conducted with time steps  $\Delta t_L$  of 0.25, 1.0, 2.0, 4.0, and 20.0 s. Shimada (1989) reported results for time steps up to 10.0 s. A 0.43 s time step and the acoustic wavespeeds shown in Table 1 were used in the water hammer analysis.

Head traces at nodes 1 and 2 are presented in Fig. 3(a) for a 300 s closure (beginning at 0 s) of the terminal reservoir valve. Although not rapid, a 300 s valve closure is still quite

fast (approximately  $15 \times 2L/a$  for the system as a whole) given the unfavorable  $\tau$ -curve. Results of the lumped ( $\Delta t_L = 4.0$  s) and water hammer analyses agree for the first 300 s leading up to full valve closure, after which the lumped analysis corresponds with the approximate mean of the water hammer analysis. The shock waves represent the rapid transformation of energy between kinetic and strain forms, a process that will gradually be dampened until the water hammer analysis converges to the lumped analysis.

Fig. 3(b) shows the edge flows during the 300 s closure experiment. The principle attribute of this valve closure algorithm is its ability to preserve continuity at nodes upstream and downstream of a valve following full closure. For example, as the flow in edge V approaches zero, the flow in edges I and IV converges to the same value; similarly, the flow in edge III after closure approaches zero while the flow in edges I and II converges to the same value. Clearly, continuity of flow is preserved both at nodes 1 and 2 following closure. This valve closure algorithm generally preserves continuity at multipipe junctions for a variety of temporal step sizes. In fact, similar observations are gleaned from 100 s closure edge flow traces for  $\Delta t_L$  of 0.25, 1.0, 2.0, and 4.0 s (Axworthy 1997).

Node 2 pressure head traces for  $\Delta t_L$  of 0.25, 1.0, 2.0, 4.0, and 20.0 s are summarized in Fig. 4 to demonstrate the sensitivity of a lumped analysis to temporal step size. If the  $\Delta t_L = 0.25$  s trace is considered an "exact" solution (i.e., most compatible to that of the water hammer analysis), it is clear

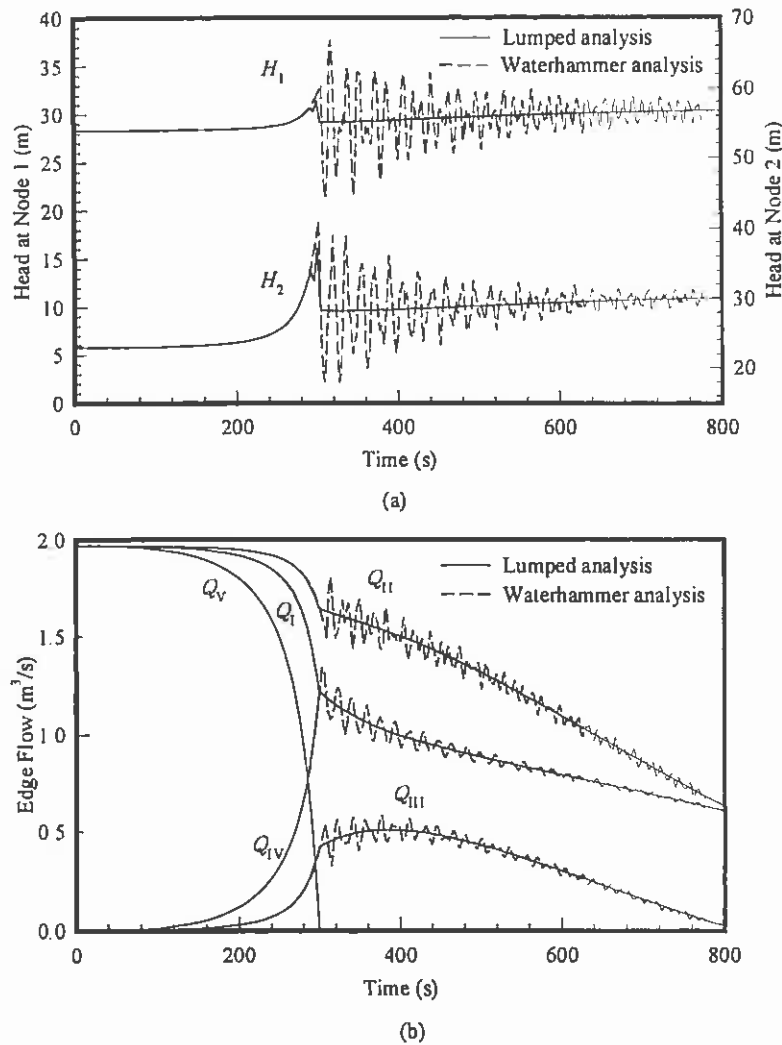


FIG. 3. Comparison of Lumped ( $\Delta t_L = 4.0$  s) and Waterhammer Analysis: (a) Nodal Heads; (b) Edge Flows for 300 s Closure of Circular Gate Valve

that accuracy diminishes with increasing temporal step size. In addition, a secondary peak, which becomes more pronounced with increasing temporal step size, develops. The "valley" associated with the peak is the result of neglecting compressibility effects in the lumped analysis and is neither initiated nor mediated by the valve closure algorithm presented in this paper. As the valve closure time increases, this secondary peak gradually disappears and compressibility effects become progressively less significant; eventually, as the closure time increases even further, even the inertial effects will disappear, and a quasi-steady model will become suitable.

Compressibility effects are near maximum just prior to closure, and since the associated discharge and inertia are small, it is not surprising that a rigid water column model exhibits a discontinuity at this point. The difficulty is even more obvious if we consider the momentum equation for unsteady incompressible flow in a single pipe containing a valve (Pickford 1969):

$$\frac{dQ}{dt} = \frac{gA}{L} \left[ H_u - H_d - RQ|Q| - \frac{Q|Q|}{(\tau E_s)^2} \right] \quad (10)$$

where  $g$  = gravitational acceleration;  $A$  = pipe cross-sectional area;  $H_u$  = upstream piezometric head;  $H_d$  = downstream piezometric head; and  $R$  = pipe frictional constant. As  $\tau \rightarrow 0$ , the valve head loss term becomes indeterminate, a difficulty that must be resolved by secondary considerations, such as considering either the relative rates of change of  $Q$  and  $\tau$  or

through compressibility effects. This mathematical difficulty arises solely from the fact that the valve closes. Thus, there is no simple resolution—one must either abandon the many advantages of the lumped water column model and use a water hammer approach, or live with the numerical difficulties and resolve them pragmatically, as is attempted here.

## CONCLUSIONS

When flow conditions in a pipe change at a moderate rate, the fluid compressibility and the conduit deformation are often insignificant, but the fluid inertia can still be important. A good approximation of these flow conditions is obtained using rigid water column theory. By considering a valve closure algorithm for slow transient, node-based, graph-theoretical models, this paper discusses only one of the many applications of the rigid water column approach. The proposed algorithm is less computationally demanding than conventional approaches, which can require significant iteration to establish nodal continuity following full valve closure. Application of the technique to a pipe network shows good preservation of nodal continuity upon and after full valve closure. In addition, comparison of results from a lumped model implementation and a water hammer model shows good nodal head agreement for slow valve closure. Moreover, both models converge to the same result as the exchange of kinetic and strain energies diminish following valve closure. Since the computational savings associated with the valve closure algorithm are potentially significant in

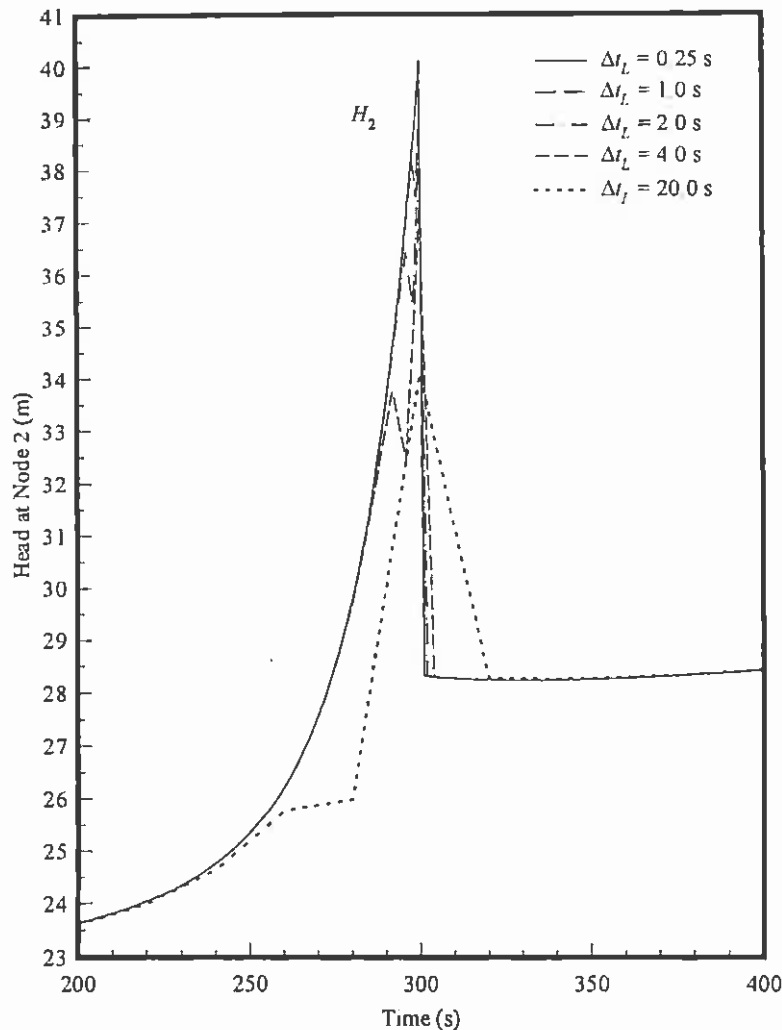


FIG. 4. Head at Node 2 for 300 s Closure of Circular Gate Valve

large network simulations, this technique is recommended for slow transient pipe network models.

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## APPENDIX II. NOTATION

The following symbols are used in this paper:

- $A$  = pipe cross-sectional area;  
 $a$  = acoustic wavespeed;

- $E_v$  = valve size parameter;  
 $\mathbf{F}(\mathbf{Q})$  = pipe friction loss vector;  
 $\mathbf{G}_1$  = incidence matrix rows corresponding to nodes having both unknown head and unknown outflow;  
 $\mathbf{G}_2$  = incidence matrix rows corresponding to nodes having unknown head and known outflow;  
 $\mathbf{G}_3$  = incidence matrix rows corresponding to nodes having known head and unknown outflow;  
 $g$  = gravitational acceleration;  
 $H_1, H_2$  = piezometric head at node 1 and node 2;  
 $\mathbf{H}_1$  = nodal head vector rows corresponding to nodes having both unknown head and unknown outflow;  
 $\mathbf{H}_3$  = nodal head vector rows corresponding to nodes having known head and unknown outflow;  
 $H_d$  = downstream piezometric head;  
 $H_v$  = valve head loss;  
 $H_u$  = upstream piezometric head;  
 $i$  = pipe index (subscript);  
 $j$  = pipe index (subscript);  
 $k$  = integer counter;  
 $L$  = pipe length;  
 $\mathbf{L}$  = pipe inertia coefficient matrix;  
 $M$  = individual pipe head difference entry in  $\mathbf{M}$  vector;  
 $\mathbf{M}$  = head difference vector;  
 $Q$  = pipe discharge;  
 $\mathbf{Q}$  = pipe flow vector;  
 $Q_v$  = discharge through valve;  
 $Q'_i = dQ_i/dt$ ;  
 $\mathbf{q}_k$  = known nodal outflow vector (head unknown);

$R$  = pipe frictional constant;  
 $\mathbf{R}$  = pipe inertia coefficient matrix;  
 $s$  = relative residual valve travel;  
 $T_c$  = total time of valve closure;  
 $t$  = elapsed time since beginning of valve closure;  
 $\mathbf{V(Q)}$  = valve loss vector;  
 $W$  = individual element inertia entry in  $\mathbf{W}$  square matrix;

$\mathbf{W}$  = coefficient matrix in part representing pipe inertia;  
 $Y$  = individual demand acceleration entry in  $\mathbf{Y}$  vector;  
 $\mathbf{Y}$  = rate of change of nodal outflow vector;  
 $\Delta t$  = time step size;  
 $\Delta t_L$  = time step size of lumped analysis;  
 $\epsilon_v$  = specified convergence tolerance; and  
 $\tau$  = nondimensional effective valve opening.